

LANDAU DIAMAGNETISM OF DEGENERATE COLLISIONAL PLASMA

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For the first time the kinetic description of Landau diamagnetism for degenerate collisional plasma is given. The correct expression for transverse electric conductivity of the quantum plasma, found by authors (see arXiv:1002.1017 [math-ph] 4 Feb 2010) is used. In work S. Dattagupta, A.M. Jayannavar and N. Kumar [Current science, V. 80, No. 7, 10 April, 2001] was discussed the important problem of dissipation (collisions) influence on Landau diamagnetism. The analysis of this problem is given with the use of exact expression for transverse conductivity of quantum plasma.

Key words: degenerate collisional plasma, magnetic susceptibility, transverse electric conductivity, Landau diamagnetism.

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1. Introduction

Magnetisation of electron gas in a weak magnetic fields compounds of two independent parts (see, for example, [1]): from the paramagnetic magnetisation connected with own (spin) magnetic momentum of electrons (*Pauli's paramagnetism*, W. Pauli, 1927) and from the diamagnetic magnetisation connected with quantization of orbital movement of electrons in a magnetic field (*Landau diamagnetism*, L. D. Landau, 1930).

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Landau diamagnetism was considered till now for a gas of the free electrons. It has been thus shown, that together with original approach developed by Landau, expression for diamagnetism of electron gas can be obtained on the basis of the kinetic approach [2].

The kinetic method gives opportunity to calculate the transverse dielectric permeability. On the basis of this quantity its possible to obtain the value of the diamagnetic response.

However such calculations till now were carried out only for collisionless case. The matter is that correct expression for the transverse dielectric permeability of quantum plasma existed till now only for gas of the free electrons. Expression known till now for the transverse dielectric permeability in a collisional case gave incorrect transition to the classical case [3]. So this expression were accordingly incorrect.

In work [4] for the first time the expression for the quantum transverse dielectric permeability of collisional degenerate plasma has been derived. The obtained in [4] expression for transverse dielectric permeability satisfies to the necessary requirements of compatibility.

Central result from [5] connects the mean orbital magnetic moment, a thermodynamic property, with the electrical resistivity, which characterizes transport properties of material. In this work was discussed the important problem of dissipation (collisions) influence on Landau diamagnetism. The analysis of this problem is given with use of exact expression of transverse conductivity of quantum plasma.

In work [6] is shown that a classical system of charged particles moving on a finite but unbounded surface (of a sphere) has a nonzero orbital diamagnetic moment which can be large. Here is considered a non-degenerate system with the degeneracy temperature much smaller than the room temperature, as in the case of a doped high-mobility semiconductor.

In the present work with use of correct expression for the trans-

verse conductivity [4] the kinetic description of Landau diamagnetism for degenerate collisional plasma for the first time is given.

2. The general expression for a magnetic susceptibility

Magnetization vector \mathbf{M} of electron plasma is connected with current density \mathbf{j} by the following expression [7]

$$\mathbf{j} = c \operatorname{rot} \mathbf{M},$$

where c is the light velocity.

Magnetization vector \mathbf{M} and a magnetic field strength $\mathbf{H} = \operatorname{rot} \mathbf{A}$ are connected by the expression

$$\mathbf{M} = \chi \mathbf{H} = \chi \operatorname{rot} \mathbf{A},$$

where χ is the magnetic susceptibility, \mathbf{A} is the vector potential.

From these two equalities for current density we have

$$\mathbf{j} = c \operatorname{rot} \mathbf{M} = c \chi \operatorname{rot} (\operatorname{rot} \mathbf{A}) = c \chi [\nabla(\nabla \mathbf{A}) - \Delta \mathbf{A}].$$

Here Δ is the Laplace operator.

Let the scalar potential is equal to zero. Vector potential we take orthogonal to the direction of a wave vector \mathbf{k} ($\mathbf{kA} = 0$) in the form of a harmonic wave

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 e^{i(\mathbf{k}\mathbf{r} - \omega t)}.$$

Such vector field is solenoidal

$$\operatorname{div} \mathbf{A} = \nabla \mathbf{A} = 0.$$

Hence, for current density we receive equality

$$\mathbf{j} = -c \chi \Delta \mathbf{A} = c \chi k^2 \mathbf{A}. \quad (1.1)$$

On the other hand, connection of electric field \mathbf{E} and vector potential \mathbf{A}

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \frac{i\omega}{c} \mathbf{A}$$

leads to the relation

$$\mathbf{j} = \sigma_{tr} \mathbf{E} = \sigma_{tr} \frac{i\omega}{c} \mathbf{A}, \quad (1.2)$$

where σ_{tr} is the transverse electric conductivity.

For our case from (1.1) and (1.2) we obtain following expression for the magnetic susceptibility

$$\chi = \frac{i\omega}{c^2 k^2} \sigma_{tr}. \quad (1.3)$$

3. Magnetic susceptibility of degenerate plasma

Landau diamagnetism in collisionless plasma is usually defined as a magnetic susceptibility in a static limit for a homogeneous external magnetic field. Thus the diamagnetism value can be found by means of (1.1) through two non-commutative limits

$$\chi_L = \lim_{k \rightarrow 0} \left[\lim_{\omega \rightarrow 0} \chi(\omega, k, \nu = 0) \right]. \quad (2.1)$$

Here ν is the effective electron collision frequency.

In collisionless plasma this expression (2.1) should lead to the known formula for Landau diamagnetism

$$\chi_L = -\frac{1}{3} \left(\frac{e\hbar}{2mc} \right)^2 \frac{p_F m}{\pi^2 \hbar^3}. \quad (2.2)$$

Here e and m are the electron charge and mass accordingly, \hbar is the Planck's constant, p_F is the electron momentum on Fermi's surfaces which is considered as spherical.

We take the Cartesian coordinates system with an axis x directed lengthways vector \mathbf{k} , and an axis y along the vector \mathbf{A} . Then expression for transverse conductivity of degenerate collisional plasma is defined by the general formula [4]

$$\sigma_{tr} = \frac{e^2 2m^3}{\omega(2\pi\hbar)^3} \int \frac{(f_F^+ - f_F^-)/\hbar + (\omega - kv_x)\delta(\mathcal{E}_F - \mathcal{E})}{\nu - i\omega + ikv_x} v_y^2 d^3v, \quad (2.3)$$

where \mathcal{E}_F is the kinetic energy of electrons on Fermi's surface, $\delta(x)$ is the Dirac's delta function,

$$\begin{aligned}\mathcal{E}_F &= \frac{p_F^2}{2m} = \frac{mv_F^2}{2}, & f_F^\pm &\equiv \Theta(\mathcal{E}_F - \mathcal{E}^\pm) \equiv \Theta^\pm, \\ \mathcal{E}^\pm &= \mathcal{E}^\pm(k) = \frac{\left(p_x \mp \frac{k\hbar}{2}\right)^2}{2m} + \frac{p_y^2 + p_z^2}{2m} = \\ &= \frac{m}{2} \left[\left(v_x \mp \frac{\hbar k}{2m}\right)^2 + v_y^2 + v_z^2 \right],\end{aligned}$$

$\Theta(x)$ is the Heaviside function,

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

We will present the formula (2.3) in the form of the sum of classical and quantum components

$$\sigma_{tr} = \sigma_{tr}^{\text{classic}} + \sigma_{tr}^{\text{quant}}, \quad (2.4)$$

where

$$\sigma_{tr}^{\text{classic}} = \frac{e^2 2m^3}{(2\pi\hbar)^3} \int \frac{\delta(\mathcal{E}_0 - \mathcal{E}) v_y^2}{\nu - i\omega + ikv_x} d^3v \quad (2.5)$$

and

$$\sigma_{tr}^{\text{quant}} = \frac{e^2 2m^3}{\omega(2\pi\hbar)^3} \int \frac{-kv_x + (\Theta^+ - \Theta^-)/\hbar}{\nu - i\omega + ikv_x} v_y^2 d^3v. \quad (2.6)$$

On the basis of equalities (2.4) – (2.6) we write down the corresponding equalities for a magnetic susceptibility

$$\chi = \chi^{\text{classic}} + \chi^{\text{quant}}, \quad (2.7)$$

where

$$\chi^{\text{classic}} = \frac{2i\omega e^2 m^3}{c^2 k^2 (2\pi\hbar)^3} \int \frac{\delta(\mathcal{E}_0 - \mathcal{E}) v_y^2}{\nu - i\omega + ikv_x} d^3v \quad (2.8)$$

and

$$\chi^{\text{quant}} = \frac{2ie^2 m^3}{c^2 k^2 (2\pi\hbar)^3} \int \frac{-kv_x + (\Theta^+ - \Theta^-)/\hbar}{\nu - i\omega + ikv_x} v_y^2 d^3v. \quad (2.9)$$

3. Transverse conductivity analysis. Diamagnetic properties of metal

Let's decompose expression (2.6) by degrees of wave number k .

The functions Θ^\pm may be presented in the form

$$f_F^\pm(k) = \Theta^\pm(k) = \Theta(\mathcal{E}_F - \mathcal{E} - \frac{\hbar^2 k^2}{8m} \pm \frac{v_x \hbar}{2} k).$$

It is clear, that

$$f_F^\pm(0) = \Theta^\pm(0) = \Theta(\mathcal{E}_F - \mathcal{E}).$$

The first derivative of Fermi — Dirac distribution function is equal to

$$\frac{\partial \Theta^\pm(k)}{\partial k} = \delta\left(\mathcal{E}_F - \mathcal{E} - \frac{\hbar^2 k^2}{8m} \pm \frac{v_x \hbar}{2} k\right) \left(-\frac{\hbar^2}{4m} k \pm \frac{v_x \hbar}{2}\right).$$

From here follows

$$\frac{\partial \Theta^\pm(0)}{\partial k} = \delta(\mathcal{E}_F - \mathcal{E}) \left(\pm \frac{v_x \hbar}{2}\right),$$

where $\delta(x)$ is the Dirac's delta function.

The second derivative of Fermi — Dirac distribution function is equal to

$$\begin{aligned} \frac{\partial^2 \Theta^\pm(k)}{\partial k^2} &= \delta'\left(\mathcal{E}_F - \mathcal{E} - \frac{\hbar^2 k^2}{8m} \pm \frac{v_x \hbar}{2} k\right) \left(-\frac{\hbar^2}{4m} k \pm \frac{v_x \hbar}{2}\right)^2 + \\ &+ \delta\left(\mathcal{E}_F - \mathcal{E} - \frac{\hbar^2 k^2}{8m} \pm \frac{v_x \hbar}{2} k\right) \left(-\frac{\hbar^2}{4m}\right). \end{aligned}$$

From here we find its value, when $k = 0$

$$\frac{\partial^2 \Theta^\pm(0)}{\partial k^2} = \delta'(\mathcal{E}_F - \mathcal{E}) \frac{v_x^2 \hbar^2}{4} - \frac{\hbar^2}{4m} \delta(\mathcal{E}_F - \mathcal{E}).$$

The third derivative of Fermi — Dirac distribution function is equal to

$$\begin{aligned} \frac{\partial^3 \Theta^\pm(k)}{\partial k^3} &= \delta''\left(\mathcal{E}_F - \mathcal{E} - \frac{\hbar^2 k^2}{8m} \pm \frac{v_x \hbar}{2} k\right) \left(-\frac{\hbar^2}{4m} k \pm \frac{v_x \hbar}{2}\right)^3 + \\ &+ 3\delta'\left(\mathcal{E}_F - \mathcal{E} - \frac{\hbar^2 k^2}{8m} \pm \frac{v_x \hbar}{2} k\right) \left(-\frac{\hbar^2}{4m} k \pm \frac{v_x \hbar}{2}\right) \left(-\frac{\hbar^2}{4m}\right). \end{aligned}$$

From here we find

$$\frac{\partial^3 \Theta^\pm(0)}{\partial k^3} = \pm \frac{v_x^3 \hbar^3}{8} \delta''(\mathcal{E}_F - \mathcal{E}) \mp \frac{3v_x \hbar^3}{8m} \delta'(\mathcal{E}_F - \mathcal{E}).$$

By means of the found derivatives we receive decomposition of the Fermi — Dirac distribution function

$$\begin{aligned} \Theta^\pm(k) = & \Theta(\mathcal{E}_F - \mathcal{E}) \pm \frac{v_x \hbar}{2} \delta(\mathcal{E}_F - \mathcal{E}) k + \\ & + \left[\frac{v_x^2 \hbar^2}{4} \delta'(\mathcal{E}_F - \mathcal{E}) - \frac{\hbar^2}{4m} \delta(\mathcal{E}_F - \mathcal{E}) \right] \frac{k^2}{2} \pm \\ & \pm \left[\frac{v_x^3 \hbar^3}{8} \delta''(\mathcal{E}_F - \mathcal{E}) - \frac{3v_x \hbar^3}{8m} \delta'(\mathcal{E}_F - \mathcal{E}) \right] \frac{k^3}{6}. \end{aligned}$$

The difference of these decompositions, divided by Planck's constant, is equal to

$$\begin{aligned} \frac{\Theta^+(k) - \Theta^-(k)}{\hbar} = & v_x \delta(\mathcal{E}_F - \mathcal{E}) k + \\ & + \left[v_x^3 \hbar^2 \delta''(\mathcal{E}_F - \mathcal{E}) - 3 \frac{v_x \hbar^2}{m} \delta'(\mathcal{E}_F - \mathcal{E}) \right] \frac{k^3}{24}. \end{aligned}$$

The second term of numerator from integrand (2.6) is equal to

$$\begin{aligned} \frac{\Theta^+(k) - \Theta^-(k)}{\hbar} - k v_x \delta(\mathcal{E}_0 - \mathcal{E}) = \\ = \frac{\hbar^2 k^3}{24} \left[v_x^3 \delta''(\mathcal{E}_0 - \mathcal{E}) - 3 \frac{v_x}{m} \delta'(\mathcal{E}_0 - \mathcal{E}) \right]. \end{aligned}$$

Now for quantum conductivity (2.6) we have the following expression

$$\sigma_{tr}^{\text{quant}} = \frac{e^2 m^3 \hbar^2 k^3}{12\omega (2\pi \hbar)^3} \int \frac{v_x^2 \delta''(\mathcal{E}_F - \mathcal{E}) - (3/m) \delta'(\mathcal{E}_F - \mathcal{E})}{\nu - i\omega + ikv_x} v_x v_y^2 d^3 v. \quad (3.1)$$

According to (2.8) we will write expression for the classical diamagnetic susceptibilities

$$\chi^{\text{classic}} = i \frac{2e^2 m^3 \omega}{(2\pi \hbar)^3 c^2 k^2} \int \frac{\delta(\mathcal{E}_0 - \mathcal{E}) v_y^2}{\nu - i\omega + ikv_x} d^3 v, \quad (3.2)$$

and according to (2.9) we will write expression for the quantum diamagnetic susceptibility

$$\chi^{\text{quant}} = i \frac{e^2 m^3 \hbar^2 k}{12(2\pi \hbar)^3 c^2} \int \frac{v_x^2 \delta''(\mathcal{E}_0 - \mathcal{E}) - (3/m) \delta'(\mathcal{E}_0 - \mathcal{E})}{\nu - i\omega + ikv_x} v_x v_y^2 d^3 v. \quad (3.3)$$

From the expression (3.2) we see, that

$$\lim_{\omega \rightarrow 0} \chi^{\text{classic}} = 0. \quad (3.4)$$

The relation (3.4) means, that magnetisation of the classical gas is equal to zero.

According to (1.3) and taking into account deduced earlier (see [4]) formulas for transverse conductivity of quantum degenerate plasma we receive the following expression for the magnetic susceptibility

$$\begin{aligned} \frac{\chi}{\chi_L} = & -\frac{3x}{q^2} \int_{-1}^1 \frac{(1-t^2)dt}{qt-z} + \frac{3}{q} \int_{-1}^1 \frac{t(1-t^2)dt}{qt-z} + \\ & + \frac{3}{4} \int_{-1}^1 \frac{(1-t^2)^2 dt}{(qt-z)^2 - q^4/4}, \end{aligned} \quad (3.5)$$

where χ_L is the value of Landau diamagnetism entered by equality (2.2).

Here for convenience dimensionless variables are entered

$$z = \frac{\omega + i\nu}{k_F v_F} = x + iy, \quad x = \frac{\omega}{k_F v_F}, \quad y = \frac{\nu}{k_F v_F}, \quad q = \frac{k}{k_F},$$

where k_F is the Fermi's wave number, $k_F = \frac{mv_F}{\hbar} = \frac{p_F}{\hbar}$, p_F is the electron momentum on Fermi's surface.

The formula (3.5) will be used for the graphic investigation of a magnetic susceptibility.

4. Landau diamagnetic susceptibility

Landau's diamagnetic susceptibility of collisionless plasma we will define the following double non-commutative limit

$$\chi_L(\nu = 0) = \lim_{k \rightarrow 0} \left[\lim_{\omega \rightarrow 0} \chi(\omega, k, \nu = 0) \right]. \quad (4.1)$$

Let's show, that with the use of the relation (3.3) the equality (4.1) leads to known Landau's formula (2.2).

Now we will consider collisionless plasma, i.e. we will put $\nu = 0$. Besides, we will consider a static limit, having put $\omega = 0$. Expression (3.3) becomes simpler thus:

$$\chi^{\text{quant}} = \frac{e^2 m^3}{96\pi^3 \hbar c^2} \int \left[v_x^2 \delta''(\mathcal{E}_F - \mathcal{E}) - \frac{3}{m} \delta'(\mathcal{E}_F - \mathcal{E}) \right] v_y^2 d^3 v. \quad (4.2)$$

Calculating integral from (4.2) in spherical system of coordinates, we obtain

$$J = \int \left[v_x^2 \delta''(\mathcal{E}_F - \mathcal{E}) - \frac{3}{m} \delta'(\mathcal{E}_F - \mathcal{E}) \right] v_y^2 d^3 v = J_1 - \frac{3}{m} J_2,$$

where

$$J_1 = \frac{4\pi}{15} \int_0^\infty v^6 \delta''(\mathcal{E}_F - \mathcal{E}) dv,$$

$$J_2 = \frac{4\pi}{3} \int_0^\infty v^4 \delta'(\mathcal{E}_F - \mathcal{E}) dv.$$

After replacement of a variable of integration $v = \sqrt{2\mathcal{E}/m}$, we have

$$J_1 = \frac{16\sqrt{2}\pi}{15m^{3/2}} \int_0^\infty \delta''(\mathcal{E}_F - \mathcal{E}) \mathcal{E}^{5/2} d\mathcal{E} = \frac{4\pi v_F}{m^3},$$

$$J_2 = \frac{8\sqrt{2}\pi}{3m^{3/2}} \int_0^\infty \delta'(\mathcal{E}_F - \mathcal{E}) \mathcal{E}^{3/2} d\mathcal{E} = \frac{4\pi v_F}{m^2}.$$

Considering that

$$J_1 - \frac{3}{m} J_2 = -\frac{8\pi v_F}{m^3},$$

we obtain expression for a magnetic susceptibility

$$\chi_L = -\frac{e^2 v_F}{12\pi^2 \hbar c^2}. \quad (4.3)$$

It is easy to check up, that expression (4.3) exactly coincides with the known Landau expression (2.2).

5. Diamagnetic susceptibility in collisional plasma

Let's consider the Landau magnetic susceptibility in collisional plasma, i.e. at $\nu \neq 0$. From the formula (3.3) it follows, that

$$\chi_L^{\text{quant}}(\omega, \nu \neq 0) = \lim_{k \rightarrow 0} \chi^{\text{quant}}(\omega, \nu, k) = 0. \quad (5.1)$$

Let's consider the diamagnetic susceptibility χ at finite values of wave vector \mathbf{k} .

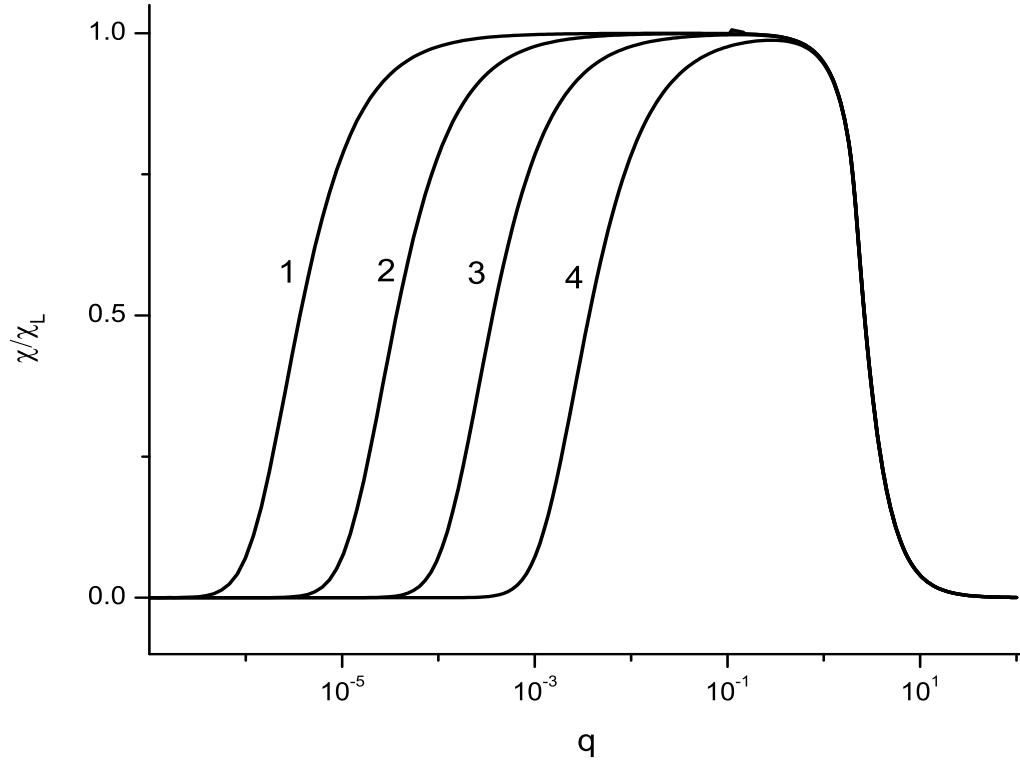


Figure 1: Diamagnetic susceptibility for the case $x = 0$ ($\omega = 0$) (a static limit), curves 1, 2, 3, 4 correspond to values of parametre $y = 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}$.

In this case expression for the diamagnetic susceptibility is given by the formula (3.5). On fig. 1 there are presented dependence plot of the dimensionless diamagnetic susceptibility (divided by Landau diamagnetic susceptibility) from the dimensionless wave numbers at various electron collisions frequencies. At $k \sim k_F$ ($q \sim 1$) the diamagnetic susceptibility coincides with Landau diamagnetism. At decreasing of wave

number the reduction of a diamagnetic susceptibility is observed up to disappearances at $k \rightarrow 0$.

To understand the mechanism of this phenomenon, we will consider a semiclassical picture of electron movement in a magnetic field.

At the finite values of quantity k the greatest contribution to the diamagnetic response bring electrons with Larmor radius orbits $\sim 1/k$. Thus frequency of their rotation on this orbit $\sim v_F k$. If collisional frequency of electrons exceeds rotation frequency, i.e. $\nu > v_F k$, than corresponding Bohr orbits collapses and restores classical picture of electrons movement. But in classical plasma diamagnetism is absent. It occurs just at $q < y$.

So our result gives answer to the question [5]: "Whether the Landau diamagnetism itself survives dissipation?" The answer is "no".

6. Conclusions

In the present work the kinetic description of Landau's diamagnetism is given for degenerate collisionless and collisional plasmas with use before the formula deduced for electric conductivity of quantum plasma. For collisionless plasmas with the help the kinetic approach the known formula of Landau diamagnetism is deduced, also it is shown, that in collisional plasma diamagnetism of degenerate electronic gas is equal to zero. Thereby the answer to a question on influence of dissipation on Landau diamagnetism is given. This question has been put in work [5].

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